HOW TEACHERS IN CHINA AND U.S. RESPOND TO STUDENT ERRORS IN SOLVING QUADRATIC EQUATIONS

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To improve mathematics achievement, students' errors should be treated as a source to stimulate their understanding of the conceptual and procedural basis of their errors. The study investigated 20 Chinese and 20 U.S. high school teachers' interpretations and responses to a student's errors in solving a quadratic equation. The teachers' responses were analyzed quantitatively and qualitatively. Analysis results show that the Chinese teachers provided more negative evaluations toward students' errors and identified more students' errors than the U.S. teachers did. Responding to students' errors, the two groups of teachers highlighted conceptual explanations targeting students' mistakes. The U.S. teachers were more likely to provide general knowledge guidance while the Chinese teachers tended to go back to basic knowledge.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge, High School Education

Introduction

Algebra has long been regarded as a critical bridge to high school mathematics. National Council of Teachers of Mathematics (NCTM, 2000) highlighted the importance of algebra to all students. The content in school algebra mainly covers two major themes: equations and functions (NCTM, 2000; Drijvers, Goddijn, & Kindt, 2010). Quadratic equations take on an important role in the high school Algebra I curriculum. From straight lines to curves it is an essential transition that requires students' conceptual understanding and computational proficiency. Prior research reveals that many students are challenged with solving quadratic equations (Vaiyavutjamai, Ellerton, & Clements, 2005; Zaslavsky, 1997). For example, Didiş, Baş, and Erbaş (2011) found that 10th graders lacked conceptual understanding of the null factor law in solving quadratic equations. Additionally, when asked to solve a quadratic equation in the form$(x-a)(x-b)=0$, many students who correctly found the solutions mistakenly held the concept that x in $(x-a)$ was equal to a, and simultaneously the x in $(x-b)$ was equal to b.

Helping students develop mathematical understanding, NCTM (2000) indicated that teachers should recognize and respond to students' errors appropriately. Students who figured out the misunderstandings under their mistakes can learn what they did not know and what they thought they knew. Rather than avoiding discussing students' errors, teachers are being called to use such errors as catalyst for stimulating reflection and exploration (Ashlock, 2006; Borasi, 1994). Taking good advantage of students' errors initiates the path of developing students' understanding of the conceptual and procedural basis of their errors.

The 2011 Trends in International Mathematics and Science Study (TIMSS) reported that both Chinese 4th graders and 8th graders outperformed their U.S. counterparts in mathematics remarkably (Provasnik et al., 2012). Teachers' knowledge has a long history of being identified as an essential factor that affects students' achievement (Ma, 1999; Hill, Rowan, & Ball, 2005). In this study, we investigated Chinese and U.S. high school algebra teachers' knowledge of interpreting and responding to students' errors in solving quadratic equations. The research questions that guided are: (1) How do Chinese and U.S. teachers interpret students' errors in solving quadratic equations?; (2) How do Chinese and U.S. teachers respond to students' errors in solving quadratic equations?; and (3) What are the similarities and differences between Chinese and U.S. teachers' knowledge of interpreting and responding to students' errors?

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Theoretical Framework

Students' Conceptual Obstacles in Solving Quadratic Equations

The methods of solving quadratic equations are introduced through factorization, the quadratic formula, and completing the square by using symbolic algorithms. Of these techniques, Didiş et al. (2011) argued that students prefer factorization since it is much faster than the other two methods. This result aligns with that from Eraslan's (2005) study. However, while applying factorization to solve quadratic equations students tended to follow the procedural rules without paying attention to the structure and conceptual meaning (Sönnerhed, 2009). As a result, they tended to make some common errors. Didiş and his colleagues (2011) summarized that when attempting to solve quadratic equations presented in a factored form, students tended to expand the parentheses to get the standard form and then re-factorize. Also, students lacked conceptual understandings of the zero-product property that they used to miss the root $x = 0$ by doing simplification. Additionally, students mistakenly tried to transfer the zero-product property into a new context, for example, to solve $(x - a)(x + b) = 12$, they simply let $x - a = 3$ and $x + b = 4$. Moreover, students used "and" rather than "or" to combine two solutions of a quadratic equation. This finding aligns with those from Ellerton and Clements (2011) that 79% of the 328 preservice middle school teachers in the study did not know that $x^2 + 6 = 0$ has no real-number solutions and many of them thought two x's in $(x - 2)(x + 3) = 0$ hold different values.

Analytical Framework

Tables 1 and 2 present the analytical frameworks utilized in the study. Peng and Luo (2009) developed a framework to analyze teachers' knowledge of students' mathematical errors (see Table 1). They identified four analytical categories for the dimension of phrases of error analysis, namely, identify, interpret, evaluate, and remediate. The levels within each dimension of teacher knowledge of students' mathematical errors are sequential and hierarchical, with progress from one level to the next, and the different levels of analysis support and complement one another by giving a holistic and structured picture of teacher knowledge of students' mathematical errors.

Table 1: Framework for Phrases of Error Analysis (Peng &Luo, 2009)

Dimension	Analytical categorization	Description
Phrases of error analysis	Identify	Knowing the existence of mathematical error
	Interpret	Interpreting the underlying rationality of mathematical error
	Evaluate	Evaluating students' levels of performance according to mathematical error
	Remediate	Presenting teaching strategy to eliminate mathematical error

Referring to the description, the phrase of remediate actually is responding to students' errors. Analyzing preservice teachers' responses to students' errors of proportional reasoning in similar rectangles, Son (2013) developed a framework to analyze teachers' responses to students' mistakes (See Table 2). According to Son (2013), conceptual knowledge is defined as the explicit or implicit understanding of the principles that govern a domain and the interrelations between pieces of knowledge in a domain. Procedural knowledge is defined as the action sequences for solving problems. Form of address signifies whether teachers deliver verbal or non-verbal information for students to hear and see (this kind of responses usually uses the very words "show" or "tell") or for students to do something and to answer questions (this kind of responses usually uses the very words "give" and "ask"). Act of communication barrier refers to the difficulties students and teachers have in communicating about students' errors. In the over-generalization category, teachers tend to provide too general an intervention that doesn't directly address students' misunderstandings. By using a Plato-and-the-slave-boy approach, teachers assume that students actually know how to solve the problem correctly but simply have forgotten. Therefore, teachers plan to ask students questions in order to help them to recall the math facts and procedures to solve problems. Returning to the basics means simply leading students to return to underlying principle. This method is regarded as either introducing more problems for students or making students forget the original problem.

Table 2: Analytical Framework for PST's Responses to Students' Mistakes (Son, 2013)

Aspect	Categories
1 Mathematical/ instructional focus	Conceptual vs. procedural
2 Form of address	Show-tell vs. give-ask
3 Pedagogical action(s)	Re-explains, suggests cognitive conflict, probes student thinking, etc.
4 Degree of student error use	Active, intermediate, or rare
5 Act of communication barrier	Over-generalization, a Plato-and-the-slave-boy approach, or a return to the basics

Methods

Twenty Chinese teachers and twenty U.S. teachers who have taught Algebra I before or are currently teaching Algebra I participated in this study. While most of the U.S. teachers hold master degrees most of the Chinese teachers have bachelor degrees. The U.S. teachers took more college level math courses than the Chinese teachers did. The group of Chinese teachers are more experienced than the group of U.S. teachers. In terms of the time that students spent on learning Algebra, it seems that Chinese students do not take as many classes as U.S. students do, but Chinese students spend more than twice of the time that U.S. students spend in doing homework. All the participants are currently teaching at high schools that have characteristics typical of each nation's public schools with respect to the students' ethnic, economic, and cultural diversity.

Figure 1 shows the main task used for this study. This problem was developed by Ellerton and Clements (2011) to test teachers' knowledge of quadratic equations. The participants were asked to analyze and respond to Amy's errors. Their written responses were coded in terms of the analytical frameworks shown in Tables 1 and 2. While analyzing the responses, we expected new categories to come out, which would optimize the existing frameworks. We first coded the participants' evaluations of the student's performance on the math topic, then examined whether the participants discovered all the student's mistakes presented in the question scenario, and finally checked whether the participants identified any underlying mathematical concepts and principles that Amy lacked of.

The participants' responses in helping Amy to correct her errors were analyzed in terms of the five aspects as elaborated in Table 2. The conceptual versus procedural distinction was utilized first, followed by the identification of pedagogical actions. After addressing these global oriented characteristics of the teachers' responses, more detailed analysis was conducted with respect to teaching approaches: form of address, degree of student error use and communication barriers. Each participant's response might be assigned more than one code within each category since more than one teaching strategy might be applied.

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Students were asked to solve $(x + 2)(2x + 5) = 0$, then to check their answer. One student,

Amy, wrote the following (line numbers have been added):

$(x + 2)(2x + 5) = 0$ *Line* 1
$\therefore 2x^2 + 5x + 4x + 10 = 0$ *Line*2
$\therefore 2x^2 + 9x + 10 = 0$ *Line*3
$\therefore (2x + 5)(x + 2) = 0$ *Line* 4
$\therefore (2x + 5) = 0 \; and \; (x + 2) = 0$ *Line* 5
$\therefore 2x = -5 \; and \; x = -2$ *Line* 6
$\therefore x = -\frac{5}{2} \; and \; x = -2$ *Line* 7

Check: Put $x = -5/2$ in $(2x + 5)$, and put $x = -2$ in $(x + 2)$.

Thus, when $x = -5/2$ and $x = -2$, $(2x + 5)(x + 2)$ is equal to 0×0 which is equal to 0. Since 0 is on the right-hand side of the original equation, it follows that $x = -5/2$ and $x = -2$ are the correct solutions.

Figure 1. Main task for the study.

Amy in Figure 1 did not have a clear understanding of the following four pieces of mathematical concepts and principles: (1) Rationale of the method of factorization; (2) Zero product property; (3) Difference between "and" and "or"; and (4) Meaning of solutions for quadratic equations. There were three mistakes from Amy's response.

- Mistake 1: Lines 2, 3, and 4 were unnecessary, since the left-side is already factored in Line 1.
- Mistake 2: In Lines 5 through 7, the word "or", and not "and", should have been used.
- Mistake 3: As for the checking process, each solution should have been substituted into both parentheses in the initial equation.

Results

Identify Students' Errors

Most of the Chinese and the U.S. teachers identified that Amy did some unproductive work. Around half of the Chinese and the U.S. teachers noticed that Amy mistakenly checked the solutions. While 80% of the Chinese teachers recognized Amy used "and" to combine the two solutions only 40% of the U.S. teachers identified it. So, the number of the Chinese teachers who found the second mistake was twice as many as that of the U.S. teachers. In addition, the Chinese teachers identified more of Amy's errors in solving the quadratic equation than the U.S. teachers did. Table 3 presents the distribution of US and Chinese teachers in identifying Amy's mistake.

Table 3: Identifications of Amy's Mistakes on Solving the Quadratic Equation

Categories	Chinese (n=20)	U.S. (n=20)
Mistake 1	14(70%)	14(70%)
Mistake 2	17(85%)	8(40%)
Mistake 3	12(60%)	11(55%)
No mistake	1(5%)	2(10%)
One mistake	3(15%)	6(30%)
Two mistakes	8(40%)	9(45%)
Three mistakes	8(40%)	3(15%)

Interpret Students' Errors

As it is shown in table 4, most of the Chinese and the U.S. teachers did not try to identify which mathematical knowledge that Amy lacked of. Among those teachers who interpreted the mathematical knowledge that Amy needed, the Chinese teachers emphasized the difference between "and" and "or" while the U.S. teachers focused on zero-product property.

Table 4: Interpretations of the Mathematical Knowledge that Amy Needed

Category	Chinese (n=20)	U.S. (n=20)
Rationale of the factoring method	2(10%)	1(5%)
Zero-product property	2(10%)	7(35%)
Differences between "and" and "or"	8(40%)	0(0%)
Meaning of solutions of quadratic equations	0(0%)	3(15%)
No interpretation	10(50%)	12(60%)
One interpretation	9(45%)	5(25%)
Two interpretations	0(0%)	3(15%)
Three interpretations	1(5%)	0(0%)

Evaluate Students' Performance

Evaluating Amy's performance, 90% of the Chinese teachers condemned Amy's performance while 10% gave a half and half comment that suggested Amy did something correct but also made mistakes. No Chinese teacher provided positive evaluations. Different from the Chinese teachers, 30% of the U.S. teachers did not evaluate Amy's overall performance. Almost half of the U.S. teachers gave half and half evaluations, whereas 15% of the teachers were positive about Amy's performance. None of the U.S. teacher gave negative evaluations. Thus, the U.S. teachers seem to be more tolerant than the Chinese teachers in front of students' errors.

Respond to Students' Errors

Around 50% of the Chinese teachers did not specifically address any mistake. 20% of the Chinese teachers demonstrated the first and the third mistakes respectively while 45% of them addressed the second mistake, that is, Amy used "and" to connect the two solutions. About one fourth of the U.S. teachers did not respond to Amy's mistakes (see Table 5). While more than fifty percent of the U.S. teachers addressed the third mistake, around 40% of them addressed the first mistake, the second mistake was neglected by most of them.

We found that the U.S. teachers differed from the Chinese teachers in terms of the number of teachers who addressed Amy's mistakes. The same number of Chinese teachers and U.S. teachers responded to two or three mistakes. In terms of Amy's three mistakes, the Chinese teachers highlighted using "or" but not "and" to connect the two solutions while the U.S. teachers emphasized how to check the solutions. Furthermore, it was found that Chinese teachers tended to address Amy's errors conceptually while the U.S. teachers favored conceptual and procedural explanations equally.

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Table 5: Mistakes Addressed by the Teachers

Category	Chinese (n=20)	U.S. (n=19)	Total (n=39)
Mistake 1	5(25%)	7(36.8%)	12(30.8%)
Mistake 2	9(45%)	3(15.8%)	12(30.8%)
Mistake 3	4(20%)	11(57.9%)	15(38.5%)
No mistake	11(55%)	5(26.3%)	16(41.0%)
One mistake	4(20%)	9(47.4%)	13(33.3%)
Two mistakes	1(5%)	3(15.8%)	4(10.3%)
Three mistakes	4(20%)	2(10.5%)	6(15.4%)

Table 6: Mathematical Knowledge Addressed by the Teachers

Category	Chinese(n=17)	U.S. (n=10)	Total(n=27)
Rationale of the factoring method	1(5.9%)	1(10%)	2(7.4%)
Zero-product property	13(76.5%)	10(100%)	23(85.2%)
Difference between "and" and "or"	7(41.2%)	1(10%)	8(29.6%)
Meaning of solutions of quadratic functions	9(53.0%)	1(10%)	10(37.0%)
One piece of knowledge	6(35.3%)	7(70%)	13(48.2%)
Two pieces of knowledge	9(52.9%)	3(30%)	12(44.4%)
Three pieces of knowledge	2(11.8%)	0(0%)	2(7.4%)

Since some teachers addressed more than one piece of conceptual knowledge, the percentage for each knowledge category in Table 6 was calculated out of 100%. As for the four pieces of mathematical knowledge which have been identified as the reasons for Amy's mistakes, most of the Chinese teachers addressed the zero-product property and around half of the Chinese teachers explained the difference between "and" and "or" and the meaning of solutions of quadratic functions. Only one Chinese teacher explained that the rationale of the factoring method was the zero-product property. Also, one U.S. teacher addressed this rationale. While all the U.S. teachers elaborated the zero-product property, the other three pieces of knowledge were overlooked by them. To conclude, the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge.

Table 7 summarizes local characteristics of the teachers' responses to Amy's errors. The Chinese teachers all applied "show and tell" strategy while some of them simultaneously asked Amy questions to likely include her in the teaching process. Almost half of the Chinese teachers did not employ Amy's mistakes in their responses while the number of the Chinese teachers who actively addressed Amy's errors and intermediately used Amy's errors are equally distributed. Additionally, the Chinese teachers tended to go back to basic knowledge.

Similar to the Chinese teachers, the U.S. teachers also emphasized "show and tell" approach. In terms of "use of student error," most of the U.S. teachers employed Amy's errors when responding to her. Moreover, they tended to hold the thought that Amy just temporarily forgot the knowledge required to solve the equation and she would perform well if they can ask questions to help her refresh the knowledge and procedures.

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Table 7: Local Characteristics of the Teachers' Responses to Amy's Errors

Aspect	Categories	Chinese(n=20)	U.S.(n=19)
Form of address	1. Show and tell	20(100%)	15(78.9%)
	2. Give and ask	7(35%)	6(31.6%)
Use of student error	1. Active use	4(20%)	7(36.8%)
	2. Intermediate use	5(25%)	4(21.1%)
	3. Rare use	11(55%)	8(42.1%)
With/Without communicative barrier	1. Over-generalization	7(35%)	5(26.3%)
	2. Plato-and-the-slave-boy	1(5%)	4(21.1%)
	3. Return to the basics	8(40%)	5(26.3%)
	4. Specific to student error	7(35%)	6(31.6%)

Discussion and Conclusions

We found that the Chinese teachers identified more of the student's mistakes than the U.S. teachers did and they are less tolerant to the student's mistakes than the U.S. teachers. Most of the teachers identified more than one of Amy's errors but they did not address all the identified errors when responding to Amy. Most of the teachers did not interpret the mathematical knowledge that Amy needed while they identifying her errors but they explained the knowledge that they believe Amy needed when responding to her. The Chinese teachers explained the mathematical knowledge conceptually and most of them demonstrated more than one piece of knowledge. In sum, the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge.

Interestingly, both the Chinese and the U.S. teachers intended to use teacher-centered pedagogical actions that highlighted "show and tell." More U.S. teachers than Chinese teachers seemed to believe that Amy simply needed help to recall all the needed mathematical knowledge so they actively used Amy's mistakes to deduce her lapses in knowledge about solving quadratic equations. The Chinese tended to go back to basic knowledge, maybe this practice is time-consuming but it is helpful for students to solve related problems correctly in the future. This study has implications to teacher educators and professional developers in both US and China.

First, both the Chinese teachers and the U.S. teachers showed the gap between the errors they identified and the errors they addressed. Since it is the errors that teachers addressed help students learn from their mistakes, teacher educators need to consider instructional interventions to help teachers develop strategies and knowledge to identify and address students' errors consistently. Second, both the Chinese teachers and the U.S. teachers applied "show and tell" approach. Teacher-centered instruction helps students to recall what they learned and provides students opportunities to learn what they missed. However, using teacher-centered instructions in front of students' errors can not probe why and how students made the errors. To learn from errors, students should know why and how they made such errors. Therefore, teacher educators need to train teachers to use multiple ways, including both teacher-centered approaches and student-centered approaches, to respond to students' errors. Third, given that the Chinese teachers outperformed the U.S. teachers in both the variety and the quantity of the addressed conceptual knowledge, professional developers may consider sessions to help in-service teachers in U.S. to construct deep conceptual understandings of certain mathematics topics that students usually are challenged by. Of relevance, teacher educators may also consider to adopt professional development sessions to help preservice teachers become sufficient in dealing with students' errors and in supporting students become mathematically competent. Last but not least, since Chinese teachers are more likely to give negative comments and less likely to employ students' errors when responding to students' errors, professional developments that help Chinese teachers build reasonable attitudes towards students' mistakes and develop flexible strategies to deal with students' errors should be considered.

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